Second-Order differential equations and price expectation model

Given the following market model with price expectations:

$$\begin{cases} Q_d = 42 - 4p(t) - 4p'(t) + p''(t) \\ Q_s = -6 + 8p(t) \\ Q_d = Q_s \end{cases}$$

Where $p(t)$ is the general price level and $p'(t) = \frac{dp}{dt}$; $p''(t) = \frac{d^2p}{dt^2}$. Find the price trajectory over time $p(t)$.

1. Determine the general solution $p(t)$.
2. Determine the particular solution of the differential equation if $p(0) = 6$ and $p'(0) = 4$.
3. Analyze stability.

Solution

1. First, we equate supply and demand:

$$42 - 4p - 4p' + p'' = -6 + 8p$$

$$-12p - 4p' + p'' = -48$$

We establish the homogeneous solution through the characteristic polynomial:

$$r^2 - 4r - 12 = 0$$

$r = -2$ and $r = 6$ are the solutions. Hence, the homogeneous solution is:

$$p_H = C_1 e^{-2t} + C_2 e^{6t}$$

Now we find the particular solution, assuming $p_c = A$. Then $p_c' = 0$ and $p_c'' = 0$

$$-12A = -48$$

$$A = 4$$

The particular solution is:

$$p_c = 4$$

The general solution is:

$$p_g = p_c + p_H = 4 + C_1 e^{-2t} + C_2 e^{6t}$$

2. Using the initial conditions to find the values of C_1 and C_2:

$$6 = 4 + C_1 + C_2$$

And

$$p_g' = -2C_1 e^{-2t} + 6C_2 e^{6t}$$

Evaluating at the initial point:

$$-2C_1 + 6C_2 = 4$$

Combining the two equations we obtain $C_1 = C_2 = 1$. Therefore, the solution is:

$$p_g = 4 + e^{-2t} + e^{6t}$$

3. Analyzing stability:

$$\lim_{t\to\infty} 4 + e^{-2t} + e^{6t} = \infty$$

Therefore, it is not stable.